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Transitional Wigner surmises from the spacing distribution of 4 × 4 matrices

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Abstract

Recently Nieminen introduced in this journal a 4 × 4 random matrix to study transitional distributions between Wigner surmises of random matrix theory. We find analytical expressions for the distributions that they obtained numerically. We also study the Ginibre-to-GOE transition.

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In a recent paper in this journal, Nieminen [1] introduced a random matrix H that transitions through all Wigner surmises. He studied the transitional distributions numerically (except GOE-to-GUE, which was done analytically). Here, we derive those results analytically and also study a new one, Ginibre-to-GOE. The relevance to physics of analytical results is that it helps in understanding system transitions induced by external parameters. An analogy is the transition between Poisson and Wigner-Dyson statistics (indicating the evolution from integrability to chaos) induced by the strength of an impurity in quantum spin chains [2]. Other work on analytical results on the spacing distribution of small dimension random matrices has been published recently [3].

For $a/2, b/2, c, d, e, f$ Gaussian distributed random variables of mean zero and variance 1, and α_j is a real parameter in the range $0 \leqslant \alpha_j \leqslant 1$:

$$H = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ c & 0 & b & 0 \\ 0 & c & 0 & b \end{pmatrix} + \mathrm{i}\alpha_1 \begin{pmatrix} 0 & 0 & d & 0 \\ 0 & 0 & 0 & -d \\ -d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 & 0 & e \\ 0 & 0 & -e & 0 \\ 0 & -e & 0 & 0 \\ e & 0 & 0 & 0 \end{pmatrix} + \mathrm{i}\alpha_3 \begin{pmatrix} 0 & 0 & 0 & f \\ 0 & 0 & f & 0 \\ 0 & -f & 0 & 0 \\ -f & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

For each $(\alpha_1\alpha_2\alpha_3)$, the eigenvalues of H are double degenerate. The near-neighbor spacing is

$$s = 2\sqrt{g^2 + c^2 + \alpha_1^2 d^2 + \alpha_2^2 e^2 + \alpha_3^2 f^2}, \tag{2}$$

where $2g = a - b$ and consequently has zero mean and variance 1.

It is shown in [1] that the probability density function of $\frac{s}{2}$ is obtained by randomly sampling an N-dimensional (the limiting cases are $N = 5$ if all α are nonzero, and $N = 2$ if all are zero) space using the following probability density function:

$$\Omega(g, c, d, e, f; \alpha_1, \alpha_2, \alpha_3) = \frac{1}{2\pi}\left[\prod_{i=1}^{M+2}\left(\frac{1}{\sqrt{2\pi}\tilde{\alpha}_i}\right)\right]\exp\left[-\frac{1}{2}\sum_{i=1}^{M+2}\left(\frac{x_i}{\tilde{\alpha}_i}\right)^2\right], \tag{3}$$

where $x_1 = g$ etcetera, $\tilde{\alpha}_1 = \tilde{\alpha}_2 = 1, \tilde{\alpha}_3 = \alpha_1, \tilde{\alpha}_4 = \alpha_2, \tilde{\alpha}_5 = \alpha_3$ (unless some of the α_i vanish as explained below) and M is the number of α that are nonzero (if $\alpha_j = 0$, the corresponding variable in equation (2) does not contribute, and the dimensionality N of the space is reduced in one unit).

For example, if no α vanishes, $M = 3$ and $\vec{x}$ is five dimensional. In that case, the coordinates can be parametrized in hyperspherical coordinates [4]:

$$\begin{cases} f = x_5 = \frac{s}{2}\cos\xi \\ e = x_4 = \frac{s}{2}\sin\xi\cos\psi \\ d = x_3 = \frac{s}{2}\sin\xi\sin\psi\cos\theta \\ c = x_2 = \frac{s}{2}\sin\xi\sin\psi\sin\theta\cos\varphi \\ g = x_1 = \frac{s}{2}\sin\xi\sin\psi\sin\theta\sin\varphi \end{cases} \qquad 0 \leqslant \varphi \leqslant 2\pi \quad 0 \leqslant \theta, \psi, \xi \leqslant \pi. \tag{4}$$

Then the desired probability density function F corresponds to all points $\vec{x}$ in a thin shell around $\frac{s}{2}$, namely $\frac{s}{2} \leqslant |\vec{x}| \leqslant \frac{s}{2} + \mathrm{d}\left(\frac{s}{2}\right)$ (where d here represents a differential, not to be confused with the parameter introduced in (1)):

$$\begin{aligned} F(s, \alpha_1, \alpha_2, \alpha_3)\,\mathrm{d}s = {} & \frac{\mathrm{d}(s/2)}{(2\pi)^{5/2}\alpha_1\alpha_2\alpha_3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\left[\left(\frac{s}{2}\right)^4\sin^3\xi\sin^2\psi\sin\theta\,\mathrm{d}\xi\,\mathrm{d}\psi\,\mathrm{d}\theta\,\mathrm{d}\varphi\right] \\ & \times\exp\left[-\left(\frac{s}{2}\right)^2\left(\sin^2\xi\sin^2\psi\sin^2\theta + \frac{\sin^2\xi\sin^2\psi\cos^2\theta}{\alpha_1^2}\right.\right. \\ & \left.\left. + \frac{\sin^2\xi\cos^2\psi}{\alpha_2^2} + \frac{\cos^2\xi}{\alpha_3^2}\right)\right]. \end{aligned} \tag{5}$$

The expression inside the first square bracket corresponds to the volume of the thin shell of width $\mathrm{d}s/2$. Similar expressions are obtained when two α are zero (integral in 3D space), and when only one α vanishes (integral in 4D space). In all cases, the integral in φ equals 2π and thus the problem reduces to performing a triple, double, or single integral.

In [1] the probability density function (pdf) was found numerically for GUE-to-Ginibre and Ginibre-to-GSE.

Pdf for GUE-to-Ginibre

This transitional case corresponds to $\alpha_1 = 1, \alpha_3 = 0, 0 \leqslant \alpha_2 \leqslant 1$. Call $\alpha_2 = \alpha$; then

$$F(s, \alpha) = \frac{s^3}{16\pi\alpha}\int_0^{\pi} \mathrm{e}^{-\frac{s^2}{8}\left(\sin^2\psi + \frac{\cos^2\psi}{\alpha^2}\right)}\sin^2\psi\,\mathrm{d}\psi, \tag{6}$$

which can be rewritten as

$$F(s, \alpha) = \frac{s^3}{16\pi\alpha}\,\mathrm{e}^{-\frac{s^2}{8\alpha^2}}\int_0^{\pi}\mathrm{e}^{\frac{s^2}{8}\left(\frac{1-\alpha^2}{\alpha^2}\right)\sin^2\psi}\sin^2\psi\,\mathrm{d}\psi. \tag{7}$$

Thus, the problem reduces to finding the integral

$$\Im(\mu) = \int_0^{\pi} e^{\mu \sin^2 \psi} \sin^2 \psi \, d\psi, \tag{8}$$

where we have defined the parameter $\mu = \frac{s^2}{8}\left(\frac{1-\alpha^2}{\alpha^2}\right)$.

It is immediate that

$$\Im(\mu) = 2\frac{d}{d\mu}\int_0^{\frac{\pi}{2}} e^{\mu \sin^2 \psi} \, d\psi = 2\frac{d}{d\mu}\int_0^{\frac{\pi}{2}} e^{\frac{\mu}{2}(1-\cos 2\psi)} \, d\psi = \frac{d}{d\mu}\left[e^{\frac{\mu}{2}}\int_0^{\pi} e^{-\frac{\mu}{2}\cos\tau} \, d\tau\right]. \tag{9}$$

The integral in the square bracket is $\pi I_0(\mu/2)$ [5]; then

$$\Im(\mu) = \frac{\pi}{2} e^{\mu/2}[I_0(\mu/2) + I_1(\mu/2)], \tag{10}$$

where I_n are modified Bessel functions [5], and we have used the properties of their derivatives.

Thus, the transitional (GUE-to-Ginibre) pdf in equation (7) is

$$F(s,\alpha) = \frac{s^3}{32\alpha} e^{-\frac{s^2}{16}\left(\frac{1+\alpha^2}{\alpha^2}\right)}\left[I_0\left(\frac{s^2}{16}\left(\frac{1-\alpha^2}{\alpha^2}\right)\right) + I_1\left(\frac{s^2}{16}\left(\frac{1-\alpha^2}{\alpha^2}\right)\right)\right]. \tag{11}$$

One is typically interested in the pdf for $\Re \equiv \frac{s}{\langle s\rangle}$, where $\langle s\rangle$ is the mean of the pdf in equation (11). To that end we need to compute

$$\langle s\rangle = \int_0^{\infty} s F(s,\alpha) \, ds. \tag{12}$$

Instead of using equation (11) directly, we return to equation (7) and write the mean explicitly as

$$\langle s\rangle = \int_0^{\infty} s \, ds \frac{s^3}{16\pi\alpha} e^{-\frac{s^2}{8\alpha^2}} \int_0^{\pi} e^{\frac{s^2}{8}\left(\frac{1-\alpha^2}{\alpha^2}\right)\sin^2\psi} \sin^2 \psi \, d\psi. \tag{13}$$

The integral in s is of the form $\int_0^{\infty} s^4 e^{-\Gamma s^2} ds$, which can be written immediately as $\frac{3\sqrt{\pi}}{8\Gamma^{5/2}}$. Thus, (13) reduces to

$$\begin{aligned}\langle s\rangle &= \frac{3\sqrt{8}\alpha^4}{2\sqrt{\pi}}\int_0^{\pi} \frac{\sin^2\psi}{[1-(1-\alpha^2)\sin^2\psi]^{5/2}} \, d\psi \\ &= \frac{3\sqrt{8}\alpha^4}{\sqrt{\pi}}\int_0^{\pi/2} \frac{\sin^2\psi}{[1-\eta\sin^2\psi]^{5/2}} \, d\psi,\end{aligned} \tag{14}$$

where we called $\eta = 1-\alpha^2$ in the last integral. Then

$$\langle s\rangle = \frac{3\sqrt{8}\alpha^4}{\sqrt{\pi}}\left\{\frac{2}{3}\frac{d}{d\eta}\int_0^{\pi/2} \frac{d\psi}{[1-\eta\sin^2\psi]^{3/2}}\right\}. \tag{15}$$

The integral in equation (15) reduces to the complete elliptic of the first kind, $E(\eta)$ [6] so that

$$\langle s\rangle = \frac{4\sqrt{2}\alpha^4}{\sqrt{\pi}}\frac{d}{d\eta}\left[\frac{E(\eta)}{1-\eta}\right], \tag{16}$$

with the notation of AMS-55 [5].

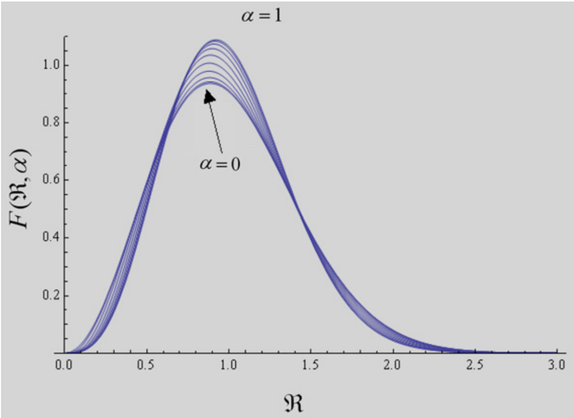

Figure 1. The transitional GUE-to-Ginibre pdfs $F(\Re, \alpha)$ have norm and mean 1. Here shown are the pdfs for values of $\alpha = \alpha_2$ ranging from 0 to 1.

Using [7], $2\eta\frac{\mathrm{d}E}{\mathrm{d}\eta} = E(\eta) - K(\eta)$, where $K(\eta)$ is the complete elliptic integral of the second kind, equation (16) becomes

$$\langle s\rangle = \frac{4\sqrt{2}\alpha^4}{\sqrt{\pi}}\frac{(1+\eta)E(\eta)-(1-\eta)K(\eta)}{2\eta(1-\eta)^2} = \frac{2\sqrt{2}}{\sqrt{\pi}}\frac{(2-\alpha^2)E(1-\alpha^2)-\alpha^2K(1-\alpha^2)}{1-\alpha^2}. \tag{17}$$

Finally, using equation (11),

$$F(\Re,\alpha) = \langle s\rangle\frac{(\langle s\rangle\Re)^3}{32\alpha}\,\mathrm{e}^{-\frac{(\langle s\rangle\Re)^2}{16}\left(\frac{1+\alpha^2}{\alpha^2}\right)} \times\left[I_0\left(\frac{(\langle s\rangle\Re)^2}{16}\left(\frac{1-\alpha^2}{\alpha^2}\right)\right)+I_1\left(\frac{(\langle s\rangle\Re)^2}{16}\left(\frac{1-\alpha^2}{\alpha^2}\right)\right)\right], \tag{18}$$

where $\langle s\rangle$ is explicitly given in equation (17).

Figure 1 shows plots of equation (18) for $0 \leqslant \alpha \leqslant 1$. As expected, the slope at the origin decreases as α increases (corresponding to the transition from $\sim\Re^2$ to $\sim\Re^3$). There is a substantial change in the value at the peak (about 20%), although the position of the peak changes only slightly.

Pdf Ginibre-to-GSE

This transitional case corresponds to $\alpha_1 = \alpha_2 = 1$, $0 \leqslant \alpha_3 \leqslant 1$. Call $\alpha_3 = \alpha$; then

$$F(s,\alpha) = \frac{s^4}{64\sqrt{2\pi}\alpha}\int_0^{\pi}\mathrm{e}^{-\frac{s^2}{8}\left(\sin^2\psi+\frac{\cos^2\psi}{\alpha^2}\right)}\sin^3\psi\,\mathrm{d}\psi. \tag{19}$$

This integral can be solved by the change of variables $x = \cos\psi$:

$$F(s,\alpha) = \frac{s^4}{32\sqrt{2\pi}\alpha}\int_0^1\mathrm{e}^{-\frac{s^2}{8}\left(1-x^2+\frac{x^2}{\alpha^2}\right)}(1-x^2)\,\mathrm{d}x = \frac{s^4\,\mathrm{e}^{-\frac{s^2}{8}}}{64\sqrt{2\pi}\alpha}\left\{\frac{\mathrm{e}^{-\mu}}{\mu}+\frac{\sqrt{\pi}\,(2\mu-1)\,\mathrm{Erf}(\sqrt{\mu})}{2\mu^{3/2}}\right\}, \tag{20}$$

where Erf is the error function and, as before, $\mu \equiv \frac{s^2}{8}\left(\frac{1-\alpha^2}{\alpha^2}\right)$.

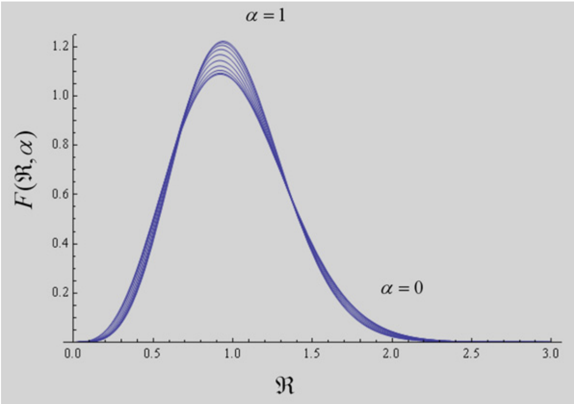

Figure 2. The transitional Ginibre-to-GSE pdfs $F(\Re, \alpha)$ have norm and mean 1. Here shown are the pdfs for values of $\alpha = \alpha_3$ ranging from 0 to 1.

To evaluate the mean value of s, we return to equation (19):

$$\langle s\rangle = \int_0^\pi \frac{\sin^3\psi\,\mathrm{d}\psi}{64\sqrt{2\pi}\alpha}\int_0^\infty s^4\,\mathrm{e}^{-\frac{s^2}{8}(\sin^2\psi+\frac{\cos^2\psi}{\alpha^2})} s\,\mathrm{d}s = \int_0^\pi \frac{\sin^3\psi\,\mathrm{d}\psi}{64\sqrt{2\pi}\alpha}\times\frac{8^3}{\left(\sin^2\psi+\frac{\cos^2\psi}{\alpha^2}\right)^3}. \tag{21}$$

Again, by writing $x = \cos\psi$, the integral becomes elementary and yields

$$\langle s\rangle = \sqrt{\frac{2}{\pi}}\left\{\frac{\alpha(3-2\alpha^2)}{1-\alpha^2}+\frac{3-4\alpha^2}{(1-\alpha^2)^{3/2}}\arccos\alpha\right\}. \tag{22}$$

Thus, from equation (20)

$$F(\Re,\alpha) = \langle s\rangle\frac{(\langle s\rangle\Re)^4\,\mathrm{e}^{-\frac{(\langle s\rangle\Re)^2}{8}}}{64\sqrt{2\pi}\alpha}\left\{\frac{\mathrm{e}^{-\hat\mu}}{\hat\mu}+\frac{\sqrt{\pi}\,(2\hat\mu-1)\,\mathrm{Erf}(\sqrt{\hat\mu})}{2\hat\mu^{3/2}}\right\}, \tag{23}$$

with $\hat\mu \equiv \frac{s^2}{8}\left(\frac{1-\alpha^2}{\alpha^2}\right)$ and $\langle s\rangle$ given by (22).

Figure 2 shows graphs of the corresponding pdfs.

Pdf Ginibre-to-GOE

This is $\vec\alpha = (0\,0\,0) \to (1\,1\,0)$, or $\vec\alpha = \alpha(1\,1\,0)$ with $0 \leqslant \alpha \leqslant 1$, such that $s = 2\sqrt{g^2+c^2+\alpha^2d^2+\alpha^2e^2}$. The space associated with equation (3) is four dimensional and the probability density becomes (after the integration in φ)

$$F(s,\alpha) = \frac{s^3}{32\pi\alpha^2}\int_0^\pi\int_0^\pi \mathrm{e}^{-\frac{s^2}{8}(\sin^2\psi\sin^2\theta+\frac{\sin^2\psi\cos^2\theta+\cos^2\psi}{\alpha^2})}\sin^2\psi\sin\theta\,\mathrm{d}\psi\,\mathrm{d}\theta. \tag{24}$$

The expression is then integrable in θ by putting $x = \cos\theta$:

$$F(s,\alpha) = \frac{s^3}{32\pi\alpha^2}\frac{2\sqrt{2\pi}\alpha}{s\sqrt{1-\alpha^2}}\int_0^\pi \mathrm{e}^{-\frac{s^2}{8\alpha^2}(\alpha^2\sin^2\psi+\cos^2\psi)}\mathrm{Erf}\left(\frac{s\sqrt{1-\alpha^2}\sin\psi}{2\sqrt{2}\alpha}\right)\sin\psi\,\mathrm{d}\psi. \tag{25}$$

We were not able to integrate the expression above exactly. However, good progress is achieved by noting that $F(s,\alpha)$ is the product of a simple, explicit function of α and s, with a

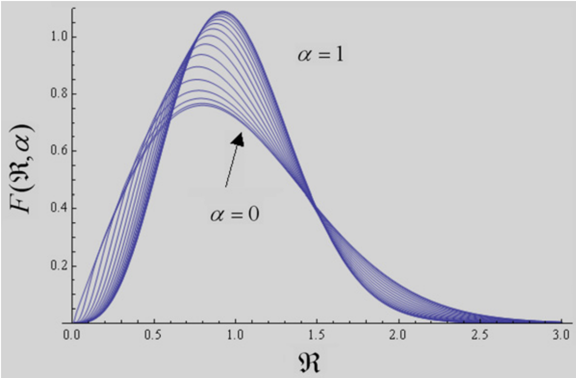

Figure 3. Transitional pdf for GOE-to-Ginibre.

parametric integral in the lumped variable $\frac{s\sqrt{1-\alpha^2}}{2\sqrt{2}\alpha}$:

$$F(s,\alpha) = \frac{s^3}{32\pi\alpha^2}\frac{2\sqrt{2\pi}\alpha \quad \mathrm{e}^{-\frac{s^2}{8}}}{s\sqrt{1-\alpha^2}}\Im(\xi), \tag{26}$$

where $\xi \equiv \frac{s\sqrt{1-\alpha^2}}{2\sqrt{2}\alpha}$, and

$$\Im(\xi) = \int_0^{\pi} \mathrm{e}^{-\xi^2\cos^2\psi}\,\mathrm{Erf}(\xi\sin\psi)\sin\psi\,\mathrm{d}\psi. \tag{27}$$

From the domains of s and α, the new variable can take values $0 \leqslant \xi < +\infty$. To simplify the study, we introduce y such that $\xi = \frac{y}{1-y}$, and $0 \leqslant y \leqslant 1$. We found an accurate representation of $\Im(y)$, to within 1% in $0 \leqslant y \leqslant 1$:

$$\Im(y) = y(1-y)\sum_n a_n T_{2n}(y), \tag{28}$$

where $T_m(y)$ are the Chebyshev polynomials of order m of the first kind and a_n are constant coefficients, $a_0 = 2.300$, $a_1 = -0.997$, $a_2 = -0.284$, $a_3 = 0.118$, $a_4 = -0.225$, $a_5 = -0.100$.

Surprisingly, the mean $\langle s\rangle$ can be obtained exactly if we return to equation (24) and first integrate the right-hand side times s from 0 to ∞. The result is

$$\langle s\rangle = \int_0^{\pi}\int_0^{\pi} \frac{3\alpha^3\sin\theta\sin^2\psi\,\mathrm{d}\psi\,\mathrm{d}\theta}{\sqrt{2\pi}[\cos^2\psi + (\cos^2\theta + \alpha^2\sin^2\theta)\sin^2\psi]^{5/2}}. \tag{29}$$

The integral in θ can then, as before, be obtained by a change of variables:

$$\langle s\rangle = \int_0^{\pi} \frac{4\sqrt{\frac{2}{\pi}}\alpha^3\sin^2\psi(2+\cos^2\psi+\alpha^2\sin^2\psi)\,\mathrm{d}\psi}{[1+\alpha^2+(1-\alpha^2)\cos 2\psi]^2} \tag{30}$$

which is

$$\langle s\rangle = \sqrt{2\pi}\frac{1+\alpha+\alpha^2}{1+\alpha}. \tag{31}$$

Figure 3 shows the pdfs for $0 \leqslant \alpha \leqslant 1$. This graph is qualitatively different from those shown in figures 1 and 2. This difference is most noticeable for values of $\Re < 1$, and it is due to the linear to cubic transition of the pdfs in the neighborhood $\Re \approx 0$.

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